Another Proof of Analyticity of Power Series

Rearranging Series and the Weierstrass Double-Series Argument

Lecture Notes

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This note gives an alternative, more conceptual proof of the analyticity of power series (the Theorem of *Power Series*, §3), due to Weierstrass: instead of the direct ε-estimate splitting the difference quotient into three pieces, it expands $f(z_0+h)$ via the binomial theorem and rearranges the resulting absolutely convergent double series. Section 1 develops the rearrangement facts needed; Section 2 gives the proof.

1 Preliminaries: Rearranging Series

Theorem (rearrangement of absolutely convergent series)

If $\sum_{n=0}^{\infty} z_n$ converges absolutely and $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then $\sum_{n=0}^{\infty} z_{\sigma(n)}$ converges absolutely, to the same sum: $\sum_n z_{\sigma(n)} = \sum_n z_n$.

Proof. First suppose $x_n \geq 0$. For any finite $F \subset \mathbb{N}$, taking $N := \max F$ gives $\sum_{n\in F} x_n \leq \sum_{n=0}^{N} x_n \leq \sum_n x_n$, so $\sup_F \sum_{n\in F} x_n \leq \sum_n x_n$; and every partial sum $\sum_{n=0}^{N} x_n$ is itself of this form (with $F = \{0,\dots,N\}$), so in fact $\sum_n x_n = \sup_F \sum_{n\in F} x_n$ – a quantity manifestly independent of any ordering of the x_n. Hence $\sum_n x_{\sigma(n)} = \sup_F \sum_{n\in F} x_n = \sum_n x_n$ for every bijection σ.

For general complex z_n with $\sum |z_n| < \infty$: writing $\operatorname{Re} z_n = \operatorname{Re}^+ z_n - \operatorname{Re}^- z_n$ and $\operatorname{Im} z_n = \operatorname{Im}^+ z_n - \operatorname{Im}^- z_n$ (positive/negative parts), each of the four nonnegative sequences $\operatorname{Re}^{\pm} z_n$, $\operatorname{Im}^{\pm} z_n$ is $\leq |z_n|$, hence sums absolutely and rearranges freely by the case above; combining the four via linearity of limits gives the same conclusion for $\sum z_n$ itself. □

Theorem (summing a double series)

Let $(b_{n,k})_{n,k\geq 0}$ be complex numbers with $\displaystyle\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |b_{n,k}| < \infty$. Then, for any bijection $\tau : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$, the series $\sum_m b_{\tau(m)}$ converges absolutely to a value independent of τ, and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{n,k} \;=\; \sum_{m} b_{\tau(m)} \;=\; \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} b_{n,k}.$$

Proof. Fix a bijection $\tau : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. Applying the rearrangement theorem above to the nonnegative terms $|b_{\tau(m)}|$ shows $\sum_m |b_{\tau(m)}| = \sum_n \sum_k |b_{n,k}| < \infty$, independent of τ; so $\sum_m b_{\tau(m)}$ converges absolutely, and by the same theorem its value doesn't depend on τ either – call it S.

Group the terms of $\sum_m b_{\tau(m)}$ "row by row": pass to the subsequence of partial sums obtained by including exactly the terms $b_{n,0},\dots,b_{n,k},\dots$ for $n = 0$, then all of row $n = 1$, and so on. A convergent series' partial sums converge, so any subsequence of partial sums – in particular one obtained by inserting parentheses – converges to the same limit S. This gives $S = \sum_n \big(\sum_k b_{n,k}\big)$. The same argument, grouping column by column instead, gives $S = \sum_k \big(\sum_n b_{n,k}\big)$. □

2 Another Proof of Analyticity of Power Series

Theorem (analyticity of power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Then f is analytic on $B(0,R)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{(termwise differentiation)}.$$

Proof (Weierstrass, via rearrangement). Let $f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$. Fix z_0 with $|z_0| < R$, and h with $0 < |h| < R - |z_0|$. By the binomial theorem, $(z_0+h)^n = \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} h^k$, so

$$f(z_0+h) = \sum_{n=0}^{\infty} a_n (z_0+h)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{n,k}, \qquad b_{n,k} := \begin{cases} \binom{n}{k} a_n z_0^{n-k} h^k, & k \le n \\ 0, & k > n. \end{cases}$$

Since

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |b_{n,k}| = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k}|h|^k = \sum_{n=0}^{\infty} |a_n| \big(|z_0| + |h|\big)^n < \infty$$

(as $|z_0| + |h| < R$), the Double Series Theorem (Theorem above) lets us sum in the other order:

$$f(z_0+h) = \sum_{k=0}^{\infty} h^k \, c_k(z_0), \qquad c_k(z_0) := \sum_{n=k}^{\infty} \binom{n}{k} a_n z_0^{n-k},$$

and moreover $\sum_{k=0}^{\infty} |c_k(z_0)||h|^k \le \sum_{n=0}^{\infty} |a_n|(|z_0|+|h|)^n < \infty$ for every such h.

In particular $c_0(z_0) = f(z_0)$ and $c_1(z_0) = \sum_{n=1}^{\infty} n a_n z_0^{n-1} = f_1(z_0)$. Fix any ρ with $0 < \rho < R - |z_0|$; then $M := \sum_{k=2}^{\infty} |c_k(z_0)| \rho^{k-2} < \infty$ (a subseries, up to the factor ρ^{-2}, of the convergent $\sum_k |c_k(z_0)|\rho^k$), and for $0 < |h| \le \rho$,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - f_1(z_0) \right| = \left| \sum_{k=2}^{\infty} c_k(z_0) h^{k-1} \right| \le |h| \sum_{k=2}^{\infty} |c_k(z_0)||h|^{k-2} \le M|h| \xrightarrow[h\to 0]{} 0.$$

Hence $f'(z_0) = \lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = f_1(z_0)$ for every $z_0 \in B(0,R)$. □

Remark. This argument shows more: re-expanded around any $z_0 \in B(0,R)$, f is itself given by the convergent power series $f(z_0+h) = \sum_k c_k(z_0) h^k$ in h (for $|h| < R - |z_0|$), so f automatically equals its own Taylor series at every point of $B(0,R)$ – the content of the *Corollary (Taylor coefficients)* in *Power Series*, §3 – with no need for a separate induction on the order of differentiation.